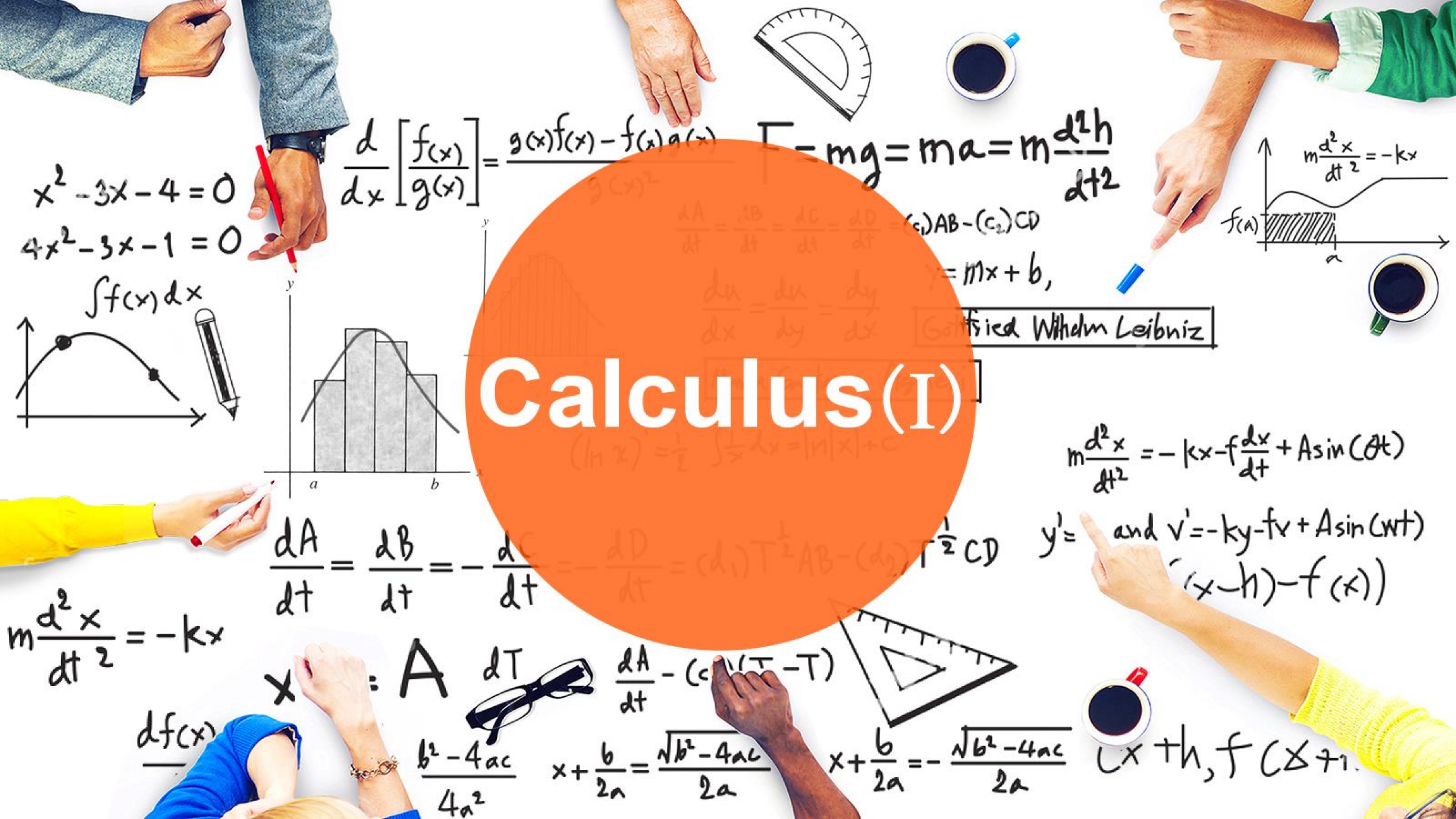


Calculus(Ⅰ)

3.2 Monotonicity and Concavity

Lecturer: Xue Deng

How to determine the monotonicity and concavity of a function?

By the signs of the first derivative and second derivative of function!

Definition of Monotonicity

Let f be defined on an interval I (open, closed ,or neither). We say that

(1) f is increasing on I if, for every pair of number x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

(2) f is deceasing on I if, for every pair of number x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

(3) f is strictly monotonic on I if it is either increasing on I or decreasing on I.

Theorem of Monotonicity

Th A: Let f be continuous on an interval I and differentiable at every interior point of I

(i) If $f'(x) > 0$ for all x interior to I, then f is increasing on I. (See Figure 1)

(ii) If $f'(x) < 0$ for all x interior to I, then f is decreasing on I. (See Figure 2)

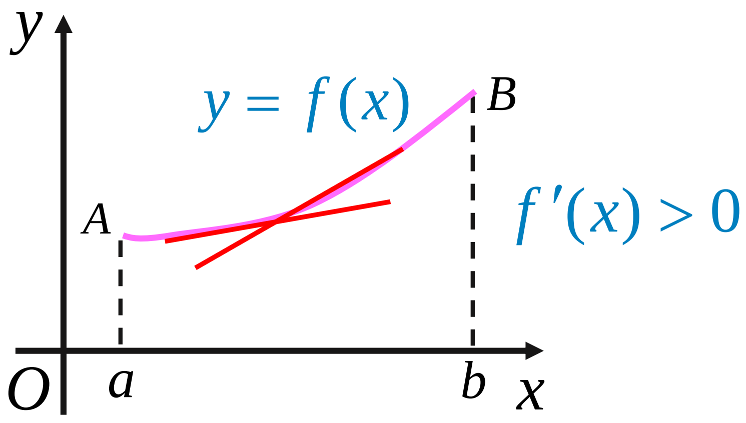


Figure 1

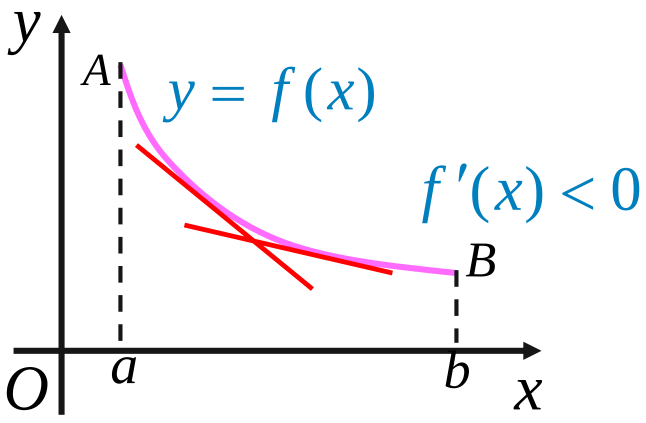


Figure 2

Definition of Concavity

Let f be differentiable on an open interval I

(1) We say that f is concave up on I if f' is increasing on I

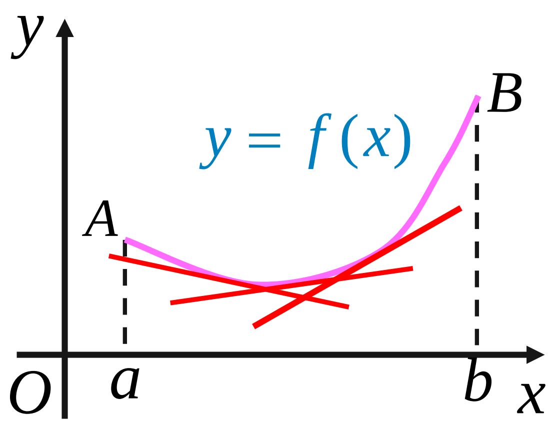
(2) We say that f is concave down on I if f' is decreasing on I

Theorem of Concavity

Th B: Let f be twice differentiable on the open interval I

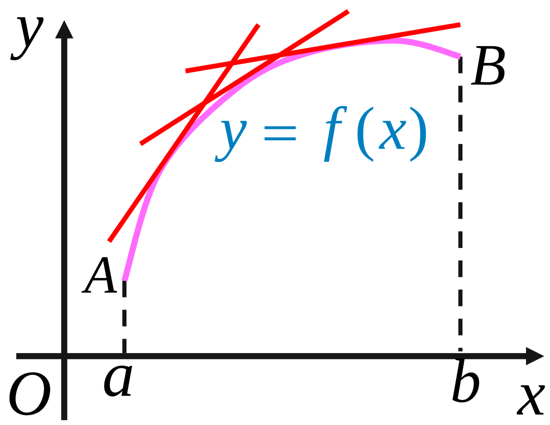
(i) If $f''(x) > 0$ for all x in I, then f is concave up on I.

(ii) If $f''(x) < 0$ for all x in I, then f is concave down on I.



$f'(x)$ increasing $f''(x) > 0$

Figure 1



$f'(x)$ decreasing $f''(x) < 0$

Figure 2

Definition of Inflection Point

Inflection Points :

Let f be continuous at c. We call $\left(c, f(c)\right)$ an **inflection point** of the graph of f if f is concave up on one side of c and concave down on the other side.

Example 1

If $y = e^x - x - 1$, find where f is increasing and where it is decreasing.

Domain is $(-\infty, +\infty)$.

$\because y' = e^x - 1$.

$\therefore (-\infty, 0)$, $y' < 0$,

$(0, +\infty)$, $y' > 0$,

$\therefore$ The function is decreasing on $(-\infty, 0]$, increasing on $[0, +\infty)$.

Example 2

If $f(x)=2x^3-9x^2+12x-3$, find where f is increasing and where it is decreasing.

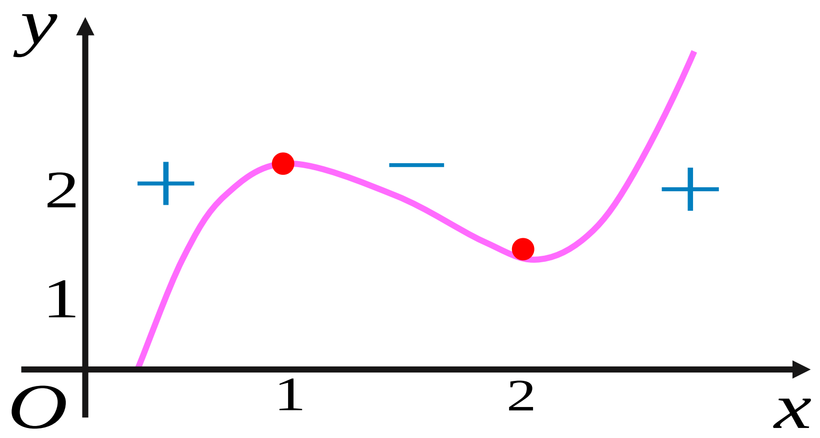
Domain is $(-\infty,+\infty)$.

$f'(x)=6x^2-18x+12$

$=6(x-1)(x-2)$

By solving the equation $f'(x)=0$,

We conclude that $x_1=1, x_2=2$.



x	$(-\infty,1)$	$(1,2)$	$(2,+\infty)$
$f'(x)$	+	−	+
$f(x)$	↗	↘	↗

∴ The function is decreasing on $[1,2]$,

increasing on $(-\infty,1]$ and $[2,+\infty)$.

Example 3

If $y = x^3$, find where y is concave up and where it is concave down.

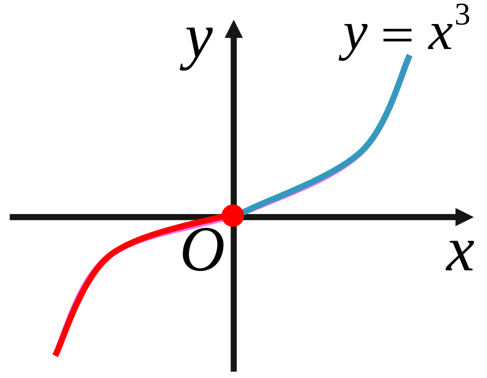
$\because y' = 3x^2,\ y'' = 6x,$

$\therefore x < 0,\quad y'' < 0,$

$\therefore y$ is concave down on $(-\infty, 0]$;

$\because x > 0,\quad y'' > 0,$

$\therefore y$ is concave up on $[0, +\infty)$.



The point $(0,0)$ is the inflection point from concaving up to concaving down

Example 4

Where is $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ increasing, decreasing, concave up, concave down?

$f'(x) = x^2 - 2x - 3 = (x+1)(x-3)$ $(x = -1 \text{ or } x = 3)$

$f''(x) = 2x - 2 = 2(x-1)$ $(x = 1)$

$(x+1)(x-3) > 0$ $\therefore f$ is increasing on $(-\infty, -1]$ and $[3, \infty)$.

$(x+1)(x-3) < 0$ $\therefore f$ is decreasing on $[-1, 3]$.

$2(x-1) > 0$ $\therefore f$ is concave up on $(1, \infty)$.

$2(x-1) < 0$ $\therefore f$ is concave down on $(-\infty, 1)$.

Example 5

Find all points of inflection of $F(x) = x^{1/3} + 2$.

$$F'(x) = \frac{1}{3x^{2/3}}$$

$$F''(x) = \frac{-2}{9x^{5/3}}$$

The second derivative, $F''(x)$, is never 0;

However, it fails to exist at $x = 0$.

The point (0,2) is an inflection point since $F''(x) > 0$ for $x < 0$ and $F''(x) < 0$ for $x > 0$.

Summary of Monotonicity and Concavity

Monotonicity Th

Let f be continuous on an interval I and differentiable at every interior point of I

(1) If $f'(x) > 0$ for all x interior to I, then f is increasing on I.

(2) If $f'(x) < 0$ for all x interior to I, then f is decreasing on I.

Concavity Th

Let f be twice differentiable on the open interval I

(1) If $f''(x) > 0$ for all x in I, then f is concave up on I.

(2) If $f''(x) < 0$ for all x in I, then f is concave down on I.

Questions and Answers

Q1: If $f(x)=2x^3-3x^2-12x+7$,

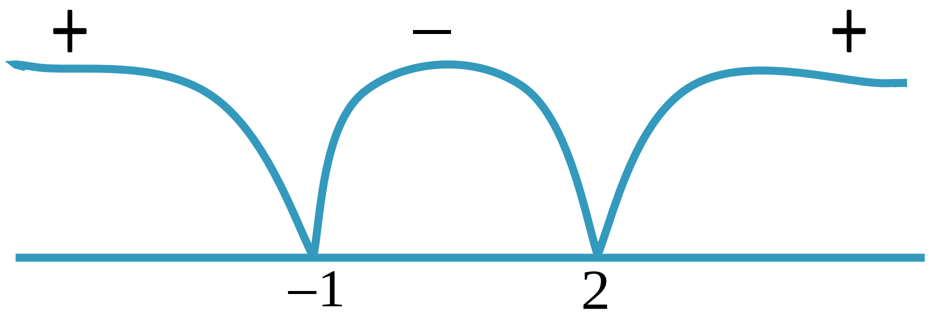
find where f is increasing and where it is decreasing.

$\because f'(x)=6x^2-6x-12=6(x^2-x-2)=6(x-2)(x+1)$,

$\therefore$ The split points are -1 and 2.

Table

x	$(-\infty,-1)$	$(-1,2)$	$(2,+\infty)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\uparrow$	$\downarrow$	$\uparrow$



$\therefore$ $f(x)$ is increasing on $(-\infty,-1]$ and $[2,+\infty)$,

$f(x)$ is decreasing on $(-1,2]$.

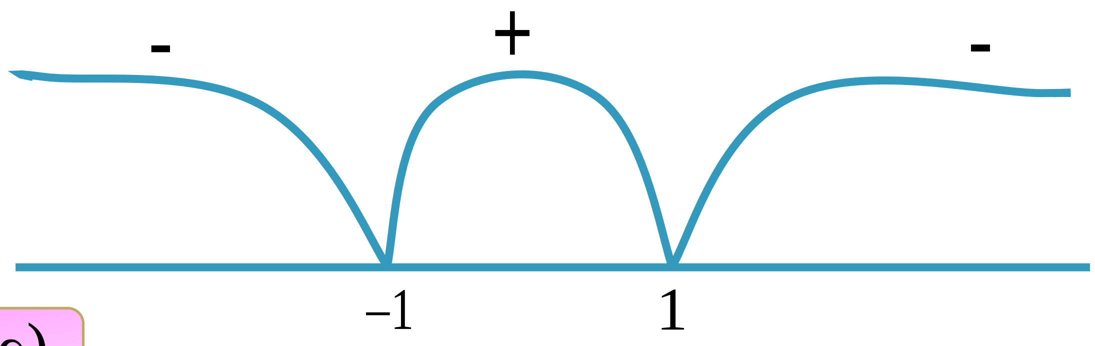
Questions and Answers

Q2: If $g(x)=\dfrac{x}{1+x^2}$, find where g is increasing and where g is decreasing.

$\because g'(x)=\dfrac{(1-x)(1+x)}{(1+x^2)^2}$ $\rightarrow$ numerator

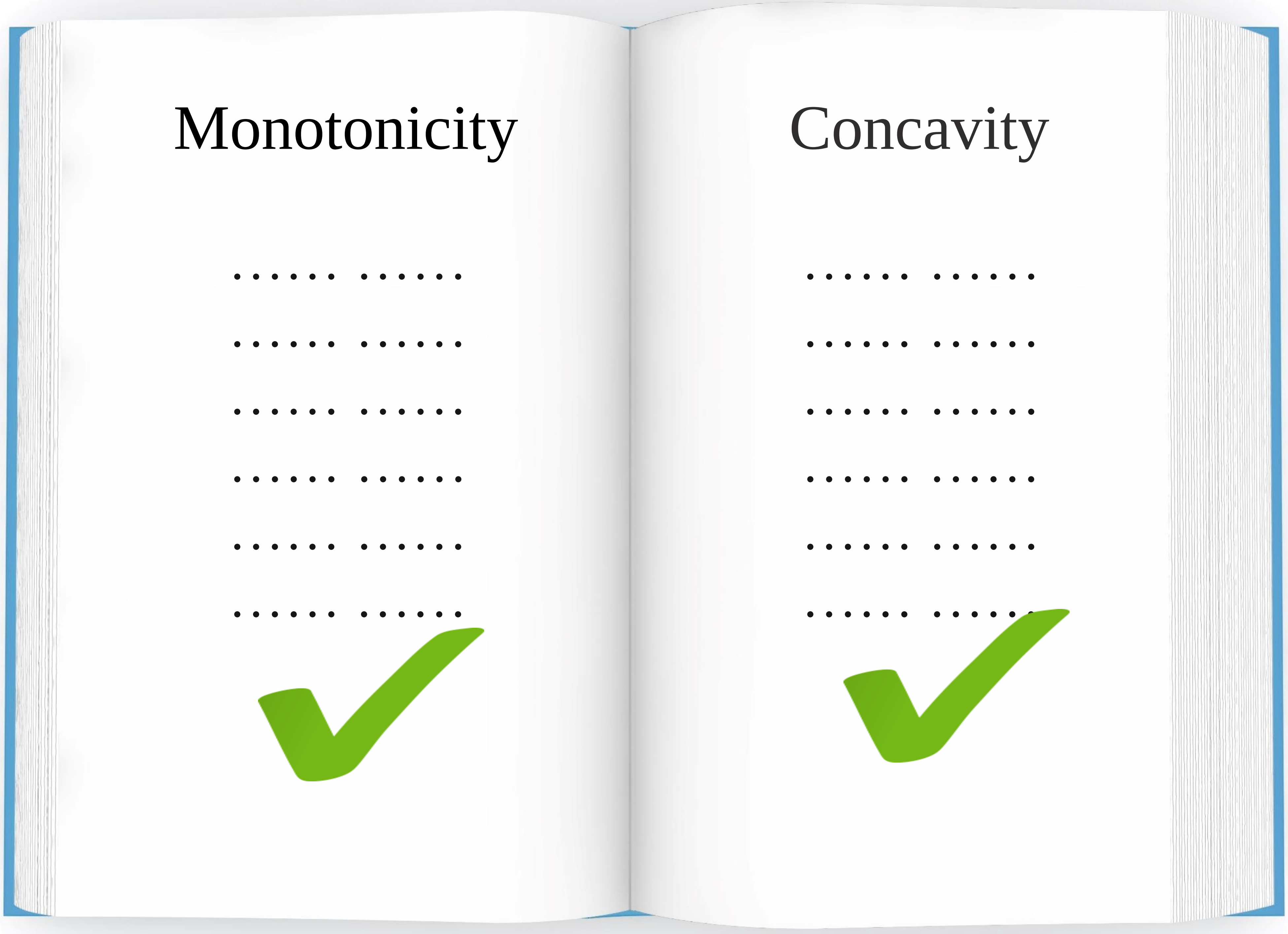
$\rightarrow$ denominator is positive at everywhere

$\therefore$ The split points are -1 and 1.



$\therefore g(x)$ is increasing on $[-1,1]$,

$g(x)$ is decreasing on $(-\infty,-1]$ and $[1,+\infty)$.


Monotonicity
Concavity